TD 6-The functor of points, fibre products

Let S be a scheme. An S-**scheme** (or R-**scheme** if $S = \mathrm{Spec}(R)$) is a scheme X together with a morphism $X \to S$. A **morphism of** S**-schemes or** S**-morphism** between T and X is a morphism of schemes $f : T \to X$ compatible with the morphisms $T \to S, X \to S$ (i.e. the obvious diagram commutes). We write $\mathrm{Hom}_S(T, X)$ or $X_S(T)$ for the set of S-morphisms between T and X, and call its elements T**-points of** X (relative to S). The **functor of points of the** S**-scheme** X is the functor $T \to X_S(T)$ (from S-schemes to sets). If $S = \mathrm{Spec}(\mathbf{Z})$, any scheme X is an S-scheme in a unique way and we write simply $X(T) = X_S(T)$. We abuse notation and write $X_{\mathrm{Spec}(R)}(T) = X_R(T)$, $X_R(\mathrm{Spec}(A)) = X_R(A)$, etc. **Whenever it is not mentioned,** X, Y **are** S**-schemes, with** S **any scheme.** Finally, if R is a ring, an R-scheme X is **locally of finite type** if X is covered by spectra of finitely generated R-algebras, and **of finite type** if moreover X is quasi-compact (i.e. **the underlying topological space** $|X|$ of X is quasi-compact).

0.1 Basic properties of the functor of points

1. a) Let R be a ring and $X = \mathrm{Spec}(R[T_1, ..., T_n]/(f_1, ..., f_r))$. Describe $X_R(T)$ for any R-scheme T. Do the same with $X = \mathrm{Spec}(R[T, 1/T])$.

   b) Construct an R-scheme X for which we have a functorial bijection between $X_R(A)$ and the set of $(a_1, ..., a_n) \in A^n$ such that $(a_1, ..., a_n) = A$, respectively between $X_R(A)$ and $\mathrm{GL}_n(A)$.

2. (**Yoneda's lemma in a special case**) a) Prove that giving an S-morphism from X to Y (X, Y being S-schemes) is equivalent to giving functorial (in T) maps of sets $X_S(T) \to Y_S(T)$ for all S-schemes T, and that it suffices to construct such maps for S-schemes T that are affine.

   b) Let $f, g \in \mathrm{Hom}_S(X, Y)$ and write $f(T), g(T) : X_S(T) \to Y_S(T)$ for the induced maps. Prove that $f = g$ if and only if $f(U_i) = g(U_i)$ for all i, where $X = \cup_i U_i$ is a fixed open covering of X.

3. a) Let U be an open subscheme of an S-scheme X (thus U is naturally an S-scheme). Prove that for all S-schemes T the natural map $U_S(T) \to X_S(T)$ is injective and identifies $U_S(T)$ with the set of S-morphisms $f : T \to X$ whose image is (set-theoretically) contained in U.

   b) Let $X = \cup_i U_i$ be an open covering of a scheme X. Prove that for any **local** ring R we have $X(R) = \cup_i U_i(R)$. Give a counter-example when R is not local.

4. (**testing surjectivity on points**) Prove that a morphism of schemes $f : X \to Y$ is surjective if and only if for any field K and any $y \in Y(K)$ there is a field extension L/K and $x \in X(L)$ whose image by $X(L) \to Y(L)$ is the image of y under $Y(K) \to Y(L)$. In particular, f is surjective if $X(K) \to Y(K)$ is surjective for all fields K. Give an example of a surjective morphism of schemes $f : X \to Y$ for which there is a field K such that $X(K) \to Y(K)$ is not surjective.

5. (a fundamental result) Let k be a field and let X be a k-scheme locally of finite type. Let X_0 be the set of closed points of X. Prove that X_0 is the set of $x \in X$ for which the extension $k(x)/k$ is finite, while $X_k(k)$ is naturally identified with the set of $x \in X$ for which $k \to k(x)$ is an isomorphism (so X_0 is identified with $X_k(k)$ when k is algebraically closed). Moreover, there is a natural bijection between the set of $\mathrm{Gal}(\bar{k}/k)$-orbits in $X_k(\bar{k})$ ($\bar{k}$ is an algebraic closure of k) and X_0.

0.2 Fibre products

If A, B, C are sets and $f : A \to C, g : B \to C$ are maps, we let $A \times_C B = \{(a, b) \in A \times B | f(a) = g(b)\}$ be the **fibre product of** A **and** B **over** C. If X, Y are S-schemes, a scheme Z is called a **fibre product of** X, Y **over** S if there are bijections of sets $Z(T) \to X(T) \times_{S(T)} Y(T)$, functorial in the scheme T.

1. a) Prove that such Z is unique up to unique isomorphism, if it exists [1]. We will denote it $Z = X \times_S Y$.

1. We will see below that it always exists!

b) Prove that there are natural morphisms (called canonical projections) $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ such that the map $f \to (p \circ f, q \circ f)$ is a bijection $(X \times_S Y)(T) \to X(T) \times_{S(T)} Y(T)$.
c) Prove that if $X \times_S Y$ exists, then $U \times_S Y$ exists for any open subscheme U of X, and is an open subscheme of $X \times_S Y$ (check that $p^{-1}(U)$ is a candidate for $U \times_S Y$).
d) Let $U \subset S$ be an open subset, and write V, W for the inverse images of U in X, Y (via the morphisms $X \to S, Y \to S$). If $X \times_S Y$ exists, prove that $V \times_U W$ exists and is an open subscheme of $X \times_S Y$.

2. We want to prove that $X \times_S Y$ always exists.
a) If $X = \text{Spec}(A), Y = \text{Spec}(B)$ and $S = \text{Spec}(C)$, prove that we can take $X \times_S Y = \text{Spec}(A \otimes_C B)$.
b) (key input) Let $X = \cup_i U_i$ be an open covering of X. Prove that if $U_i \times_S Y$ exist for all i, then $X \times_S Y$ exists. **Hint** : glue the $U_i \times_S Y$'s along suitable open subschemes, obtained using $(U_i \cap U_j) \times_S Y$.
c) Let $S = \cup_i S_i$ be an open covering of S and X_i, Y_i the inverse images of S_i in X_i, Y_i. If $X_i \times_{S_i} Y_i$ exist for all i, prove that $X \times_S Y$ exists and has an open covering by the $X_i \times_{S_i} Y_i$. Conclude!
3. Show that the category of schemes has products, i.e. for any schemes X, Y there is a scheme $X \times Y$ such that $(X \times Y)(T) = X(T) \times Y(T)$ functorially in T. Letting $\mathbf{A}^n = \text{Spec}(\mathbf{Z}[T_1, ..., T_n])$, check that $\mathbf{A}^n \times \mathbf{A}^m = \mathbf{A}^{n+m}$, but that $\mathbf{P}^n \times \mathbf{P}^m$ is not isomorphic to $\mathbf{P}^{n+m}$ for $n, m \geq 1$ (here $\mathbf{P}^n$ is the projective space over $\mathbf{Z}$). **Hint** : for the last part count the number of $\mathbf{F}_q$-points!
4. (**fibres of a morphism**) Let $f : X \to S$ be a morphism of schemes. If $s \in S$ recall that there is a canonical morphism $\text{Spec}(k(s)) \to S$, so we can define $X_s = X \times_S \text{Spec}(k(s))$. Prove that there is a natural homeomorphism $|X_s| \to f^{-1}(s)$ (so $f^{-1}(s)$ is endowed with a natural structure of $k(s)$-scheme).
5. (more difficult) Let $f : X \to S, g : Y \to S$ be S-schemes and let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the two projections. Prove that p, q induce a natural surjective continuous map

$$\pi : |X \times_S Y| \to |X| \times_{|S|} |Y| := \{(x, y) \in |X| \times |Y| | f(x) = g(y)\}$$

and $\pi^{-1}(x, y)$ is homeomorphic to $\text{Spec}(k(x) \otimes_{k(s)} k(y))$, where $s = f(x) = g(y) \in S$.

0.3 Base change I

If X is an S-scheme and $S' \to S$ is a morphism, the **base change of** X **by** $S' \to S$ is the S'-scheme $X_{(S')} := X \times_S S'$. We often write $X_{S'}$ for $X_{(S')}$ and $X \otimes_R R'$ for $X_{(S')}$ when $S = \text{Spec}(R)$, $S' = \text{Spec}(R')$.

1. a) Prove that for any S'-scheme T we have a natural bijection $X_S(T) = (X_{(S')})_{S'}(T)$ (we see T as S-scheme via $T \to S' \to S$).
b) If $X \to Z$, $Y \to Z$ are S-morphisms of S-schemes, prove that $(X \times_Z Y)_{(S')} = X_{(S')} \times_{Z_{(S')}} Y_{(S')}$.
c) Prove that if $f : X \to Y$ is an S-morphism, then f induces a canonical S'-morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$, called the **base change of** f **by** $S' \to S$. Prove that if f is surjective, then so is $f_{(S')}$, but this is false if surjective is replaced by injective or bijective.
2. Let $X = \text{Spec}(\mathbf{Q}[U, V]/(U^2 + V^2 - 1))$ and $Y = \text{Spec}(\mathbf{Q}[U, V]/(U^2 + V^2 + 1))$. Prove that X and Y are not isomorphic, but that $X \otimes_{\mathbf{Q}} \mathbf{Q}(i)$ and $Y \otimes_{\mathbf{Q}} \mathbf{Q}(i)$ are isomorphic!
3. Give an example of a scheme X over a field k such that X is connected (resp. irreducible, resp. integral) but there is a finite extension k'/k such that $X \otimes_k k'$ is no longer connected (resp....).
4. (hard) Let $f : X \to S$ be a morphism of schemes. Prove that the following statements are equivalent, in which case we say that f is **radiciel** :
a) $f_{(S')} : X_{(S')} \to S'$ is injective for all morphisms $S' \to S$.
b) f is injective and $k(f(x)) \to k(x)$ is purely inseparable for all $x \in X$.
c) $X(K) \to S(K)$ is injective for all fields K.

0.4 Dimension theory II-globalization

We globalize here the results proved for affine schemes in the previous exercise sheet. We fix a field k.

1. Suppose that X is irreducible and of finite type over k, let η be its generic point.
a) Prove that $\dim X$ is the transcendence degree of $k(\eta)$ over k. Deduce that $\dim X = \dim U$ for any nonempty subset U of X.
b) Prove that all maximal chains of closed irreducible subsets of X have the same length.
c) Prove that $\dim(O_{X,x}) = \dim X$ for any closed point $x \in X$.

2. a) Prove that if X, Y are nonempty k-schemes locally of finite type, then $\dim X \times_k Y = \dim X + \dim Y$.

   b) Prove that if X is a k-scheme locally of finite type, then for all field extensions K we have $\dim(X \otimes_k K) = \dim X$ (recall that $X \otimes_k K = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$).

0.5 Change of the base field

This exercise is fairly challenging. You may want to take for granted that all results below hold when all schemes involved are spectra of fields (then they reduce to-often nontrivial-questions in the theory of fields). You will also need the results in the exercise concerning Chevalley's theorem. We fix a field k. If X is a k-scheme and K/k is an extension of k, we write $X_K = X \otimes_k K = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. By convention, all schemes below are k-schemes. A field extension L/k is called separable if $L \otimes_k M$ is reduced for all extensions M/k.

1. a) Prove that if X, Y are k-schemes, then the natural projection $p : X \times_k Y \to X$ is surjective and open.

   b) Deduce that if C is an irreducible component of $X \times_k Y$, then the closure of $p(C)$ is an irreducible component of X, and all irreducible components of X arise this way. Moreover, for any connected component C of $X \times_k Y$, $p(C)$ is contained in a unique connected component of X, and any connected component of X is obtained in this way.

2. Let P be one of : irreducible, connected, reduced, integral. We say that X is **geometrically** P if X_K has P for all K.

   a) Prove that if $X \times_k Y$ has P, then X has P (thus if X_K has P then X has P). Prove that the converse fails, by giving an explicit example in each case.

   b) Suppose that X is geometrically P. Prove that $X \times_k Y$ has P for all k-schemes Y.

   c) Prove that if X and Y are geometrically P, then so is $X \times_k Y$.

3. Let $X = \mathrm{Spec}(\mathbf{Q}[T, S]/(T^2 - 2S^2))$. Prove that X is integral, geometrically reduced and geometrically connected, but not geometrically integral!

4. Let X, Y be k-schemes locally of finite type and $f, g : X \to Y$ two k-morphisms. Suppose that X is geometrically reduced and there is K/k algebraically closed such that f, g induce the same maps $X_k(K) \to Y_k(K)$. Prove that $f = g$.

5. a) Prove that if k is perfect, any reduced k-scheme is geometrically reduced.

   b) Prove that if X is a reduced k-scheme, then X_K is reduced for any separable extension K/k. If X is irreducible (resp. connected) and k is separably closed in K, then X_K is irreducible (resp. connected). If X is integral and K/k is separable and k is algebraically closed in K, then X_K is integral.

   c) Suppose that X is a connected k-scheme such that $X(k) \neq \emptyset$. Prove that X is geometrically connected.